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Pre-University Students' Errors in Integration of Rational Functions and Implications for Classroom Teaching

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This paper reports on students' errors in performing integration of rational functions, a topic of calculus in the pre-university mathematics classrooms. Generally the errors could be classified as those due to the students' weak algebraic concepts and their lack of understanding of the concept of integration. With the students' inability to link integration to differentiation, these errors could not be detected or rectified. From a deeper perspective, these errors were due to a lack of deep mathematical thinking when the students learnt calculus. This paper also presents the implications of the findings of this study in relation to the classroom teaching of mathematics. It is hoped that the articulation of students' errors and the implications could provide guidance for classroom teachers and prompt further research into students' errors and misconceptions in calculus concepts.

Introduction

Studies have shown that students generally have poor comprehension of the mathematical meaning of what they have learnt in calculus, for example, the concept of a limit and its meaning in calculus (Davis & Vinner, 1986), the Fundamental Theorem of Calculus, and Riemann sums and their relation to graphs (Judson & Nishimori, 2005). The concept of limit is not easily grasped by students as the common use associated with the word 'limit' may conflict with its mathematical use in calculus (Davis & Vinner, 1986; Tall & Vinner, 1981). This conflict could hinder students from achieving a deeper comprehension and application of calculus (Burn, 2005).

Key words: Calculus; Integration; Rational functions; Students' errors

Other studies indicate that students' difficulties in calculus also stem from their difficulties in dealing with functions and algebra (Judson & Nishimori, 2005). While many of the students in Judson and Nishimori's study could carry out the procedures required to find and use derivatives in sketching graphs of functions, almost all the students had a poor understanding of functions, which had led to misconceptions in solving problems, especially in the application of differentiation and in integration.

White and Mitchelmore (1996) found in their study that students' poor understanding of variables and symbols seemed to be a major source of difficulties in calculus. They suggested that students should spend more time using algebra and working with variables to reach a clearer understanding of variables before attempting calculus. They further suggested that remediation in calculus may not be as helpful as remediation in algebra. Usiskin (2003) proposed that students may fare better in calculus if they had been given early and repeated exposure to concepts such as function and limit and that students may benefit from more exposure to algebra before they attempt calculus.

This brief body of research seems to indicate that calculus is generally difficult for students who either have difficulties grasping the calculus concepts, or who lack sufficient mathematical background in functions, variables and symbols, and algebra in general.

This paper reports on a study of pre-university students' errors in calculus, in particular, their errors in performing integration of rational functions. The implications of this study to the teaching of pre-university mathematics are also discussed.

Literature Review

According to Sfard (1991), mathematical notions can be thought of operationally as processes or structurally as objects, and dual conceptions that cannot be separated as both are necessary to develop algebraic understanding. Sfard (1991) stated that operational processes naturally precede structural conceptions in algebraic development so most learners achieve operational mastery first. Since advanced mathematical constructs are abstract and not easily accessible to the senses, learners stumble over mastering structural conceptions. Sfard (1991) believed that when learners manage to master structural conceptions, these learners would be able to better organise their learning and recall. These learners would be able to switch easily between the operational mode and structural mode of thinking. Such students would be more effective than those without such organisations in solving algebra problems.

As an extension of Sfard (1991)'s idea on integration, consider $\int (6-5x)^{\frac{1}{2}} dx$, $\int x(x^2+7)^{-6} dx$ and $\int x(x^2+7) dx$. Instead of having to recall two separate formulas, $\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{n+1} + c$ and $\int f'(x)[f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1} + c$, it may be easier and more economical for students if they could recognise that the integration items fit $\int f'(x)[f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1} + c$ so that they only need to recall only one idea applicable for all three items. Further, it would also be more efficient for students to recognise that $\int x(x^2+7)dx$ fits $\int f'(x)[f(x)]^n dx = \frac{[f(x)]^{n+1}}{n+1} + c$ so that they need not multiply to obtain $\int (x^3+7x)dx$ first.

In terms of cognitive information processing, there are two main considerations: (1) how do learners understand and recall concepts in integration; and (2) what connections are formed in learners' minds as they attempt to make sense of integration?

Skemp (1986) defined abstracting as "an activity by which we become aware of similarities in our experiences" (p. 21), concept as the end-product of abstraction, and schema as a conceptual structure. A schema is a framework for inter-relating different elements of information into one conceptual unit, and for use according to its similarity to a given situation (Norman, Gentner, & Stevens, 1976). According to Skemp, new learning depends on the availability of a suitable schema and effective learning occurs when it is schematically organised rather than memorised.

Fischbein (1999) described successful thinking and the recalling process in mathematical reasoning as having a structural schema which unifies a big-picture concept or principle with a specific programme of action. He suggested that, ideally, learners should be able to identify enough similarities in a given problem to match those in previous problems. Successful learners should then be able to choose a sequence of specific procedures to react efficiently with the given problem.

Both Skemp (1986) and Fischbein (1999) argue that it is crucial for learners to organise the mathematics they learn into a workable form. Therefore, in the context of integration, it would seem that effective learning should occur when learners recall the operational processes of differentiation well and then go on to form appropriate schema which link differentiation to integration. This idea of fitting rules with patterns applies when, for example, learners work on the differentiation of to integrate rather than on integrating each of the terms of the polynomial expansion of separately.

Students' errors in calculus could also be due to their poor foundation in algebra (White and Mitchelmore, 1996; Usiskin, 2003). For example, Kirshner and Awtry (2004) suggested that errors occur in algebra because of students' wrong interpretation of algebraic objects. These wrong interpretations appeared to have resulted from the visual presentation of the algebraic objects. The algebraic rules and patterns may not have been explicitly stored in one's mind but may have developed not only from learning or practising explicit rules but also from recognising visual patterns on the printed page.

Bereiter (1991) suggested that rules provide checks on permissibility of actions, e.g. algebraic rules, or subgoals that advance problem solution. Consequently, the explicit principles may sometimes be comprehensible only after students have acquired the competence that the rules formalise.

Applying this in relation to integrating rational functions, it suggests that students may need to work with rules first, before they are shown a larger framework of integration formulas. Students need to specifically know algebra (e.g. long division of polynomials and laws of indices) before being able to identify templates or formulas to complete the integration.

The purpose of this study was to identify errors that students in a Singapore pre-university institution make in performing integration on rational functions, and the influence of understanding of algebra on their performance in integration. In particular, the following research questions were addressed:

1. What errors did the students make in integrating rational functions?
2. What algebraic errors could have caused the students to wrongly carry out integration on rational functions?

The implications of this study to the teaching of calculus in the pre-university mathematics classrooms are then discussed.

Methodology

Sleeman's Study

The method used in this study was modelled after Sleeman's (1986) study. Sleeman first administered a test to 24 students. He then collated answers according to common errors seen, and the students were then selected for interviews based on the errors they had made. Sleeman began his interview sessions with a list of tasks based on pupils' individual errors but generated different tasks as the interviews proceeded. The aim of doing so was to establish pupils' patterns of behaviour and identify their solving strategies. Our study modelled Sleeman's because one of the aims of the study was to identify students' errors. However, the methodology was modified to include only those students who were willing to be interviewed. In this study, a test was administered to collect students' written errors and interviews were used to determine students' thinking processes in producing erroneous computational steps or answers.

The interviews conducted in this study did not include having students carry out additional integration tasks. Instead, the students in this study were interviewed about their test responses. Moreover, the students in this study are older students (age 17 to 19) and hence more vocal. Thus additional tasks were replaced by more intensive interviews. The interviewees were regularly reminded that no "correct" answers were expected from them and that the purpose of the interviews was to further understand their errors and misconceptions, so that subsequently teachers could better help them cope with their difficulties in calculus.

The Method

The participants for this study were identified on voluntary basis after the entire level of Year 2 students (age 18 to 19) had taken a common test (see Figure 1) on integration. The test was conducted after the completion of the entire chapter on integration. The content material for integration can be found in the standard A-Level textbook (for example, Perkins and Perkins, 1998). Hence, the participants consisted of a small group of 20 students from one Year Two class (coded as Class 2A, the enrolment of which was 23 students) taught by one Mathematics teacher and five students from another Class (coded as Class 2B, the enrolment of which was 15 students). Both classes were taught by the same mathematics teacher.

Since the study was conducted near to the year-end examination, the participants were concerned with the amount of time taken up by the study. Hence, most of the 25 participants were unwilling to participate in the interviews; only the five students from 2B were willing to be interviewed.

The questions for the common test on integration for the entire level of Year 2 students are shown in Figure 1.

Test 6			
Time: 35 minutes			Marks: ___ / 35
1.	(i)	Find	$\int \frac{x^2+7}{x+1}\,dx$
	(ii)	Find	$\int \frac{x+2}{9+16x^2}\,dx$
	(iii)	Find	$\int x^3 \ln(5x)\,dx$
	(iv)	Find	$\int \frac{1}{\sqrt{x(8-x)}}\,dx$

Figure 1. Common test on integration conducted for the entire Year 2 level.

From the common test on integration (Figure 1), the main students' errors were identified. The four common errors that the Year 2 students made can be classified under four main categories:

1. The students did not include the constant of integration in finding an indefinite integral;
2. In integrating the indefinite integral of the rational function $\int \frac{ax+b}{cx+d}\,dx$, the students ignored the constant term of the numerator and simply evaluated $\int \frac{ax}{cx+d}\,dx$;
3. The students integrated the x-term in the numerator separately to obtain an extra term, as illustrated in the example $\int \frac{x}{3^2+(4x)^2}\,dx = \frac{x^2}{2}\left(\frac{1}{3}\tan^{-1}\left(\frac{4x}{3}\right)\right)+c$

4. The students completely ignored the algebraic expression in the numerator and evaluated the remaining expression, as illustrated in the example $\int \frac{x^2+7}{x+1}\,dx = \ln|x+1| + c$

Based on the four categories of common errors among the entire level of Year 2 students, a set of four study tests (sets A, B, C and D) was prepared to further investigate the errors. The items for the study test are shown in Figure 2.

SET A Integrate each of the following with respect to x.

(1) $\int \frac{2}{x}\,dx$ (2) $\int (2x-3)^2\,dx$

(3) $\int (6-x)^{1/2}\,dx$ (4) $\int x(x^2+7)\,dx$

(5) $\int \frac{4}{x+3}\,dx$ (6) $\int \frac{x-1}{x}\,dx$

(7) $\int \frac{x}{x-1}\,dx$

SET B Integrate each of the following with respect to x.

(1) $\int \frac{1}{2x}\,dx$ (2) $\int (2x-3)^{21}\,dx$

(3) $\int (6-5x)^{1/2}\,dx$ (4) $\int x(x^2+7)^{-6}\,dx$

(5) $\int \frac{4x}{x+3}\,dx$ (6) $\int \frac{x^2-1}{x^2}\,dx$

(7) $\int \frac{x^2}{x^2+1}\,dx$

SET C Integrate each of the following with respect to x.

(1) $\int \frac{1}{2x+7}\,dx$ (2) $\int \frac{x}{(x^2+7)^2}\,dx$

(3) $\int \frac{x+3}{x^2+16}\,dx$

SET D Integrate each of the following with respect to x.

(1) $\int \frac{x+3}{x^2+7}\,dx$ (2) $\int \frac{x+3}{(x^2+16)^2}\,dx$

Figure 2. The four sets of study test items (Sets A, B, C and D).

The set of four study tests was administered to twenty students from class 2A and the five students from another class 2B.

Each student was given all four sets of study tests at the same time, with each set having two to seven items. Altogether, there were 19 items in the four sets. The students were given 40 minutes to complete all four sets of the test. Once they had completed one set, they were instructed to work on the next set and not to refer to the previous set. This was to reduce the possibility of students gleaning hints or clues from one set to complete another set. Students had to show the relevant computational steps with their final answers. The students were not allowed to use any formula sheets or formula lists. They were also not allowed to refer to their notes or books during the test.

The five students from Class 2B were coded as L, U, N, Z and S. Students N and S were interviewed together within a week of sitting the test. L and Z were interviewed together a week later, and U was interviewed later in the same week as L and Z. All three interviews were conducted on different dates within two weeks after the test.

The main limitations of this study were the small non-random sample size and the small number of items. Further, as the interviews were carried out on three different dates, the students could have discussed the interview with each other. The students were strongly encouraged not to communicate the content of the interview with the other students, since the main objective of this interview was for the researcher to have the most accurate understanding of the students' learning difficulties in integration. It should also be mentioned that in carrying out this study, the students were not given the formula sheet on the integration formulas. Thus, some of the errors in the integration questions could also be due to memory failure.

Description of the Instrument

The test items were set in order of difficulty within each set and to allow for comparison of errors and integration techniques within each set and across sets. Items numbered 1 in sets A, B and C were meant to be easier questions for students so that they could work on more accessible items before attempting more difficult items within each set. These test items were also meant to gauge whether individual students had some idea of how to integrate simple rational functions.

These items were separated into the four different sets A, B, C and D to reduce the chances of students using clues found in earlier test items to help them solve latter items.

Results

Table 1 shows the number of students and the number of correct answers they achieved out of the 19 test items.

Table 1
Number of Correct Answers (Grouped Data)

Number of correct answers	Number of students from class 2A	Number of students from class 2B	Total number of students	Percentage of students
0 to 9	4	5	9	36%
10-19	16	0	16	64%
Total	20	5	25	100%

First Research Question: What Errors Did the Students Make in Integrating Rational Functions?

The summary of the most common errors which were consistently noted among the 25 students, either individually or across students is tabulated in Table 2, together with the number of students who committed the error with a sample of the error type.

Second Research Question: What Algebraic Errors Could Have Caused the Students to Wrongly Carry Out Integration on Rational Functions?

In this section the algebraic errors which could have caused errors in integrating rational functions are discussed. This involves an analysis of some individual students' work. It appears that many of the algebraic errors pertain more to the students' ability in algebraic manipulation rather than with calculus per se. An analysis of the error patterns and comments from relevant interviews follow.

Table 2
Errors Made by Students in Integrating Rational Functions

	Type of errors	No. of students	Sample of error of this type
1.	Wrong placement of constant term	2	$\int \frac{2}{x}dx = \frac{1}{2}\ln x + c$
2.	Wrong placement of coefficient of x	6	$\int \frac{1}{2x+7}dx = 2\ln\|2x+7\| + c$
3.	Ignored external term in x	2	$\int x(x^2+7)^{-6}dx = \frac{(x^2+7)^{-5}}{-5} + c$
4.	Ignored coefficient of x	17	$\int \frac{1}{2x+7}dx = \ln\|2x+7\| + c$
5.	Introduced unnecessary $\frac{x^2}{2}$ term into the integration	9	$\int x(x^2+7)^{-6}dx = \frac{x^2}{2}.\frac{(x^2+7)^{-5}}{-5} + c$
6.	Ignored numerator in integration	13	$\int \frac{4x}{x+3}dx = \ln\|x+3\| + c$
7.	Performed long division incorrectly	3	$\int \frac{x}{x-1}dx = \int 1 - \frac{1}{x}dx = x - \ln\|x\| + c$
8.	Moved variable outside the integration sign	2	$\int \frac{x}{x-1}dx = x\int \frac{1}{x-1}dx = x\ln\|x-1\| + c$
9.	Wrong multiplication of rational functions	3	$\int x(x^2+7)^{-6}dx = \int (x^3+7x)^{-6}dx$
10.	Adjusted index wrongly as a multiple	3	$\int (6-x)^{1/2}dx = \frac{1}{2}\frac{(6-x)^{3/2}}{-3/2}$

Algebraic Error: The "Disregarded x-term"

N had ignored the term in x completely in $\int \frac{4x}{x+3}dx$. She treated the following items the same way she treated $\int \frac{4}{x+3}dx$, all yielding logarithmic functions as integrals. N had performed the integration of $\int \frac{x-1}{x}dx$, $\int \frac{x}{x-1}dx$,

$\int \frac{x}{(x^2+7)^2}dx$, $\int \frac{x^2-1}{x^2}dx$ and $\int \frac{x^2}{x^2+1}dx$ in the same way; she had treated all such rational functions as though they were of the form $\frac{\text{constant}}{\text{function of } x}$.

Z could integrate functions involving adjusting constants. For more complicated integrands, she either integrated the term in x, or omitted doing the question completely. When asked, she said that she knew that the items $\int \frac{4}{x+3}dx$ and $\int \frac{4x}{x+3}dx$ were different but could not remember what she was supposed to do, and did not attempt to carry out the integration.

Algebraic Error: Wrong Division

Three different students had carried out long division on $\frac{x}{x-1}$ incorrectly, writing $\int \frac{x}{x-1}dx = \int 1-\frac{1}{x}dx$, $\int \frac{x}{x-1}dx = \int 1-x\,dx$ and $\int \frac{x}{x-1}dx = \int x-1\,dx$.

These students omitted showing any computational steps as to how they had carried out the division. During the interview, Z was asked why she had attempted long division before carrying out the integration for $\int \frac{x}{x-1}dx$ but had not attempted the same method for $\int \frac{4x}{x+3}dx$. Z replied "I can't differentiate. I think something needs [to be] change[d]." And after she had looked at her computation, she said, "I think something is wrong with the division." However, she was unsure what was wrong with her division. She was then asked why she omitted computing $\int \frac{4x}{x+3}dx$. She said "I know these two are different" with reference to $\int \frac{4}{x+3}dx$ and $\int \frac{4x}{x+3}dx$. But she could not carry out the integration $\int \frac{4x}{x+3}dx$. She was then directed to consider her computation for $\int \frac{x}{x-1}dx$ and asked whether the same approach could be used for both $\int \frac{x}{x-1}dx$ and $\int \frac{4x}{x+3}dx$. Z then said, "Is it?" She still not could carry out the long division. The other three students (coded as L, U and S) also omitted $\int \frac{4x}{x+3}dx$.

Algebraic Error: Moving the x-term Outside the Integration

Two students took the x-term outside the integration sign, writing

$$\int \frac{x}{x-1}dx = x\int \frac{1}{x-1}dx \text{ and } x^2\int \frac{1}{x^2+1}dx = x^2 \tan^{-1} x \ ,$$

which meant that they did not consider the x-term in the numerator to be a variable to be integrated.

Algebraic Error: Multiplying the x-term Inside the Brackets

The student L consistently treated $x(x + 7)^{-n}$, where n is an integer, in the same way as $x(x^2 + 7)$. She had not noticed the need to consider that the expression did not fit the algebraic rule $a^m \times b^m = (ab)^m$.

Discussion

Some students in this study appear not to have managed to learn integration rules or formula. Some had formed a misconceived framework to manage integration or poor recall of appropriate formulas. Some had been obstructed by poor algebraic skills.

Furthermore, as evident from their work, most students had not related integration to differentiation at all to check the correctness of their integration. Rather, the students seemed to have memorised a large range of formulas and procedures to be used for special types of integrands. The result was that they either could not recall the exact formula or were confused by the many formulas. Hence they could not carry out integration successfully. These students obviously had not formed any consistently useful schema to relate rational functions to the appropriate differentiated forms.

To phase out the possibility of incorrect memory and poor recall of the appropriate integration formulas (as described in the section on the limitation of this study), it is suggested that the researchers could have provided the students with the use of the relevant integration formula sheet.

Implications for Classroom Teaching

One possible suggestion for teaching integration as the inverse of differentiation could be to link the steps in differentiation to the reverse steps in integration using 'opposing' words and 'backward steps' as shown in Table 3. For each of the four categories, a worked example can be included for demonstration on how the integration formula is applied. Cognitive scientists and researchers who study cognitive load have shown the usefulness of including worked examples (Sweller & Chandler, 1994; Sweller, 1994). The use of worked examples can be seen as one kind of scaffolding which assist the students with instructional activities (Brush & Saye, 2001; Woolfolk, 2001).

Table 3
Recommended Chart for Teaching Integration as the Reverse Process of Differentiation

Differentiation	Integration
$\frac{d}{dx}\left(\left(ax+b\right)^n\right) = n.(ax+b)^{n-1}.(a)$ Step 1: Multiply by the index n. Step 2: Reduce the index by one. Step 3: Apply Chain Rule, i.e. differentiate $(ax+b)$ to get a, and multiply a to the expression. E.g. $\frac{d}{dx}(2x-3)^3 = 3(2x-3)^2(2) = 6(2x-3)^2$	$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$ Step 1: Apply Chain Rule, i.e. differentiate $(ax+b)$ to get a, and divide the expression by a. Step 2: Increase the index by one. Step 3: Divide by the index $n+1$. E.g. $\int (2x-3)^2\,dx = \frac{(2x-3)^3}{3(2)} + C$
$\frac{d}{dx}\left(f(x)^n\right) = n.f(x)^{n-1}.f'(x)$ Step 1: Multiply by n. Step 2: Reduce the index by one. Step 3: Apply Chain Rule to $f(x)$ and multiply $f'(x)$ to the end of the expression. E.g. $\frac{d}{dx}\left(e^{2x}+2\right)^3 = 3\left(e^{2x}+2\right)^2(2e^{2x}) = 6e^{2x}\left(e^{2x}+2\right)^2$	$\int f'(x)\left[f(x)\right]^n dx = \frac{\left[f(x)\right]^{n+1}}{n+1} + c$ Step 1: Apply Chain Rule to $f(x)$, adjust to write $f'(x)$ at the front of the expression. Step 2: Increase the index by one. Step 3: Divide by $n+1$. E.g. $\int e^{2x}\left(e^{2x}+2\right)^2 dx = \int 2e^{2x}\left(e^{2x}+2\right)^2 dx$ $= \frac{\left(e^{2x}+2\right)^3}{3} + C$

In this study, the students had been taught by their teacher that they could check whether they had carried out the correct integration by differentiating their answers and cross-checking the result with the integrand. However, in the interviews carried out none of the five students said that they checked their work in this way. From this lack of checking by differentiating answers it could be inferred that students did not specifically relate that they were doing the inverse of differentiation when they integrated functions.

One suggested recommendation is that checking the correctness of integration by using differentiation should be modelled in class by the teacher. Time should be allowed in tests for students to carry this out. Perhaps the questions could have specified this requirement; some of the marks in the test could be allocated for completion of both integration and differentiation.

The method of teaching integration by comparing it with differentiation (as illustrated by Table 3), together with constant reminders by teachers that students should differentiate answers to cross-check, may help students form more useable schemas and reduce their dependence on recalling special formulas for special integrands. In short, a comparison table (such as Figure 3) after each sub-section in the chapter of integration should be used to link the relevant differentiation and integration forms.

In addition, a summary of a 'big-picture' organisation of integrating polynomials and rational functions should be introduced for students. Students should be constantly reminded to carry out the inverse operations in integration and to compare them to differentiation. This may help students form a 'big picture' view of integration rather than making them recall the many different formulas.

Anecdotal evidence from mathematics classrooms show that the main attitude of students in learning mathematics is that to finish solving all the given problems was their main concern. Many students have little confidence in checking the correctness in their solutions. Thus, this 'big-picture' organisation will also help to reinforce this aspect of monitoring their own mathematical thinking, and to engage them in deeper understanding of mathematical concepts.

On the other hand, the procedural aspect of mathematics should also not be overlooked. Making students practise the formulas before guiding them to see the larger framework fits in with Sfard's idea that students achieve operational mastery before structural conceptions (Sfard, 1991). Also, applying rules are important in the early stages of learning new skills (Bereiter, 1991). Hence, combining Sfard and Bereiter's ideas, it may be good for students if they were to spend more time practising different formulas first, then have teachers guide them to fit the different formulas into a larger framework to aid them in understanding the mathematical concepts and relate these concepts.

As described in the earlier sections, many of the students' errors in integration stem from their weakness in algebra concepts. This should alert the teachers that the pre-calculus mathematics topics, especially concepts on algebra, should be taught more carefully.

Conclusion

Some students had approached integration with a poor background in algebra which could have caused them to misread the integration or misinterpret symbols and to carry out algebraic work such as indices and long division of polynomials incorrectly. For some students, the poor linkage between differentiation and integration meant that they could not check whether they had used appropriate methods or whether they had used the correct formula in integration.

The results of this study could provide some insight into classroom teaching of calculus, and hopefully could spur further research into students' errors in other calculus concepts.

From a deeper perspective, the errors and misconceptions in calculus in this study is clearly due to a lack of deep mathematical thinking in learning calculus. During the school years, students have rarely been given much opportunity or encouragement to think mathematically. Thus, it is not a surprise that students learn calculus (and all other mathematics topics as well) instrumentally and teachers teach calculus (and other topics) with the main objective to prepare their students for the national examinations. Perhaps a more effective teaching could take change at a deeper process, in which students unlearn their habit of superficial learning, and teachers their usual approach of procedural teaching. More focus should be made on the relational understanding of the mathematical concepts. However, this would be a long journey and a major challenge for the education system, in view of the current result-oriented school curriculum. Hopefully, this paper could encourage educators to take a deeper thought on how students can be engaged to learn and think mathematically, and be involved in 'deep learning' of concepts in calculus.

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